

Quasi-exact Solvability of Planar Dirac Electron in Coulomb and Magnetic Fields

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Abstract

The Dirac equation for an electron in two spatial dimensions in the Coulomb and homogeneous magnetic fields is a physical example of quasi-exactly solvable systems. This model, however, does not belong to the classes based on the algebra $sl(2)$ which underlies most one-dimensional and effectively one-dimensional quasi-exactly solvable systems. In this paper we demonstrate that the quasi-exactly solvable differential equation possesses a hidden $osp(2, 2)$ superalgebra.

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1 Introduction.

It is well known that exactly solvable models play an important role in many fields of physics. However, exactly-solvable systems are very rare in physics. Recently, in quantum mechanics a new type of spectral problems, which is intermediate to exactly solvable ones and non solvable ones, has been found [[1]-[7]]. For this new class of spectral problems, the so-called quasi-exactly solvable (QES) models, it is possible to determine algebraically a part of the spectrum but not the whole spectrum. Such quasi-exact solvability is usually due to some hidden Lie-algebraic structures. More precisely, a QES Hamiltonian can be reduced to a quadratic combination of the generators of a Lie group with finite-dimensional representations.

The first physical example of QES model is the system of two electrons moving in an external oscillator potential discussed in [8, 9]. Later, several physical QES models were discovered, which include the two-dimensional Schrödinger [10], the Klein-Gordon [11], and the Dirac equation [12, 13] of an electron moving in an attractive/repulsive Coulomb field and a homogeneous magnetic field. More recently, the Pauli and the Dirac equation minimally coupled to magnetic fields [14], and Dirac equation of neutral particles with non-minimal electromagnetic couplings [15] were also shown to be QES.

It turns out that the system of two electrons moving in an external oscillator potential and those of an electron moving in an Coulomb field and a homogeneous magnetic field mentioned above, with the exception of the Dirac case, share the same underlying structure that made them QES. In [16] it was demonstrated that these systems are governed essentially by the same basic equation, which is QES owing to the existence of a hidden $sl(2)$ algebraic structure. This algebraic structure was first realized by Turbiner for the case of two electrons in an oscillator potential [17]. For the Dirac case, on the other hand, it had been shown the quasi-exact solvability of this system is not related to the $sl(2)$ algebra [13]. At that time, however, it was not known to us what algebraic structure the Dirac system possesses. In this paper we would like to show that the relevant symmetry of this system is the $osp(2, 2)$ superalgebra.

2 The Dirac Equation

In $2 + 1$ dimension, the Dirac equation for an electron minimally coupled to an external electromagnetic field has the form (we set $c = \hbar = 1$)

$$(i\partial_t - H_D)\Psi(t, \mathbf{r}) = 0, \quad (1)$$

where

$$H_D = \sigma_1 P_2 - \sigma_2 P_1 + \sigma_3 m - eA^0 \quad (2)$$

is the Dirac Hamiltonian, σ_k ($k = 1, 2, 3$) are the Pauli matrices, $P_k = -i\partial_k + eA_k$ is the operator of generalized momentum of the electron, A_μ the vector potential of the external electromagnetic field, m the rest mass of the electron, and $-e$ ($e > 0$) is its electric charge. The Dirac wave function $\Psi(t, \mathbf{r})$ is a two-component function. For an external Coulomb field and a constant homogeneous magnetic field $B > 0$ along the z direction, the potential A_μ are given by

$$A^0(r) = Ze/r \ (e > 0), \quad A_x = -By/2, \quad A_y = Bx/2 \quad (3)$$

in the symmetric gauge. The wave function is assumed to have the form

$$\Psi(t, \mathbf{x}) = \frac{1}{\sqrt{r}} \exp(-iEt) \psi_l(r, \varphi), \quad (4)$$

where E is the energy of the electron, and

$$\psi_l(r, \varphi) = \begin{pmatrix} F(r)e^{il\varphi} \\ G(r)e^{i(l+1)\varphi} \end{pmatrix} \quad (5)$$

with integral number l . The function $\psi_l(r, \varphi)$ is an eigenfunction of the conserved total angular momentum $J_z = L_z + S_z = -i\partial/\partial\varphi + \sigma_3/2$ with eigenvalue $j = l + 1/2$. It should be reminded that l is not a good quantum number. Only the eigenvalues j of the conserved total angular momentum J_z are physically meaningful.

Putting Eq.(4) and (5) into (1), and taking into account of the equations

$$P_x \pm iP_y = -ie^{\pm i\varphi} \left(\frac{\partial}{\partial r} \pm \left(\frac{i}{r} \frac{\partial}{\partial \varphi} - \frac{eBr}{2} \right) \right), \quad (6)$$

we obtain

$$\frac{dF}{dr} - \left(\frac{l + \frac{1}{2}}{r} + \frac{eBr}{2} \right) F + \left(E + m + \frac{Z\alpha}{r} \right) G = 0, \quad (7)$$

$$\frac{dG}{dr} + \left(\frac{l + \frac{1}{2}}{r} + \frac{eBr}{2} \right) G - \left(E - m + \frac{Z\alpha}{r} \right) F = 0, \quad (8)$$

where $\alpha \equiv e^2 = 1/137$ is the fine structure constant. In a strong magnetic field the asymptotic solutions of $F(r)$ and $G(r)$ have the forms $\exp(-eBr^2/4)$ at large r , and r^γ with $\gamma = \sqrt{(l + 1/2)^2 - (Z\alpha)^2}$ for small r . One must have $Z\alpha < 1/2$, otherwise the wave function will oscillate as $r \rightarrow 0$ when $l = 0$ and $l = -1$. With these asymptotic factors, we now write

$$F(r) = r^\gamma \exp(-eBr^2/4) Q(r), \quad G(r) = r^\gamma \exp(-eBr^2/4) P(r). \quad (9)$$

Substituting Eq.(9) into Eq.(7) and (8) and eliminating $P(r)$ from the coupled equations, we obtain

$$\left\{ \frac{d^2}{dr^2} + \left[\frac{2\gamma}{r} - eBr + \frac{Z\alpha/r^2}{E + m + Z\alpha/r} \right] \frac{d}{dr} + E^2 - m^2 \right. \\ \left. + \frac{2EZ\alpha}{r} + \frac{l + \frac{1}{2}}{r^2} - \frac{\gamma}{r^2} - eB(\Gamma + 1) \right. \\ \left. + \frac{Z\alpha/r^2}{E + m + Z\alpha/r} \left[\frac{\gamma}{r} - eBr - \frac{l + 1/2}{r} \right] \right\} Q(r) = 0, \quad (10)$$

where $\Gamma = l + 1/2 + \gamma$. Once $Q(r)$ is solved, the form of $P(r)$ is obtainable from Eqs.(7) and (9). If we let $x = r/l_B$, $l_B = 1/\sqrt{eB}$, Eq.(10) becomes

$$\left\{ \frac{d^2}{dx^2} + \left[\frac{2\gamma}{x} - x + \frac{Z\alpha}{x((E + m)l_B x + Z\alpha)} \right] \frac{d}{dx} \right. \\ \left. + (E^2 - m^2)l_B^2 + \frac{2EZl_B\alpha}{x} + \frac{(l + 1/2 - \gamma)}{x^2} - (\Gamma + 1) \right. \\ \left. - \frac{Z\alpha(l + 1/2 - \gamma)}{x^2[(E + m)l_B x + Z\alpha]} - \frac{Z\alpha}{(E + m)l_B x + Z\alpha} \right\} Q(x) = 0. \quad (11)$$

Eq.(11) can be rewritten as

$$\left\{ \frac{d^2}{dx^2} + \left[\frac{2\beta}{x} - x - \frac{1}{x + x_0} \right] \frac{d}{dx} + \epsilon + \frac{b}{x} - \frac{c}{x + x_0} \right\} Q(x) = 0. \quad (12)$$

Here $\beta = \gamma + 1/2$, $x_0 = Z\alpha/[(E + m)l_B]$, $\epsilon = (E^2 - m^2)l_B^2 - (\Gamma + 1)$, $b = b_0 + L/x_0$, $b_0 = 2EZ\alpha l_B$, $L = (l + 1/2 - \gamma)$, and $c = x_0 + L/x_0$. The energy E is determined once

the values of ϵ and x_0 are known. The corresponding value of the magnetic field B is then obtainable from the expression $l_B = Z\alpha/[(E + m)x_0]$. Solution of ϵ and x_0 is achieved in [13] by means of the Bethe ansatz equations. There it was shown that Eq.(12) is QES when ϵ is a non-negative integer, i.e. $\epsilon = n$, $n = 0, 1, 2, \dots$. In this case, the function $Q(x)$ is a polynomial of degree n .

If we eliminate $Q(x)$ from Eqs.(7) and (8) instead, we will obtain a second differential equation of $P(x)$:

$$\left\{ \frac{d^2}{dx^2} + \left[\frac{2\beta}{x} - x - \frac{1}{x + x'_0} \right] \frac{d}{dx} + \epsilon' + \frac{b'}{x} - \frac{c'}{x + x'_0} \right\} P(x) = 0 , \quad (13)$$

with $x'_0 = Z\alpha/[(E - m)l_B]$, $\epsilon' = \epsilon + 1$, $b' = b_0 + c'$, and $c' = -\Gamma/x'_0$. Other parameters are as defined previously. It is obvious that Eq.(13) is in the same form as Eq.(12), and hence is also QES. In [13] it was shown that Eq.(12) and (13) gave the same QES spectrum when

$$\begin{aligned} \epsilon = n \quad , \quad \epsilon' = n + 1 \quad , \quad n = 0, 1, 2, \dots \quad , \\ b' - c' &= b - c + x_0 \quad , \\ x'_0 x_0 &= \frac{(Z\alpha)^2}{\Gamma + n + 1} \quad . \end{aligned} \quad (14)$$

The result that $\epsilon = n$ and $\epsilon' = n + 1$ implies that the degree of the polynomial $P(x)$ is of one order higher than that of $Q(x)$, i.e. when $\epsilon = n$, $Q(x)$ and $P(x)$ are of degree n and $n + 1$, respectively.

In [13] it was also shown that the quasi-exact solvability of Eqs.(12) and (13) is not due to the $sl(2)$ Lie-algebra, which is responsible for the quasi-exact solvability of most one-dimensional and effectively one-dimensional systems. We now want to show that the relevant algebra is indeed $osp(2, 2)$. In what follows, we shall present a differential representation of this algebra, and then use it to demonstrate how $osp(2, 2)$ underlies the QES structure of our Dirac system.

3 A Differential Representation of $osp(2, 2)$

The superalgebra $osp(2, 2)$ is characterized by four bosonic generators $T^{\pm, 0}$, J and four fermionic generators $Q_{1,2}$, $\bar{Q}_{1,2}$. These generators satisfy the commutation and anti-commutation relations [4, 7]

$$[T^0, T^{\pm}] = \pm T^{\pm} \quad , \quad [T^+, T^-] = -2T^0 \quad , \quad [J, T^{\alpha}] = 0 \quad , \quad \alpha = 0, +, - \quad ,$$

$$\{Q_1, \bar{Q}_2\} = -T^- \quad , \quad \{Q_2, \bar{Q}_1\} = T^+ \quad ,$$

$$\frac{1}{2}(\{\bar{Q}_1, Q_1\} + \{\bar{Q}_2, Q_2\}) = J \quad , \quad \frac{1}{2}(\{\bar{Q}_1, Q_1\} - \{\bar{Q}_2, Q_2\}) = T^0 \quad ,$$

$$[Q_1, T^+] = Q_2 \quad , \quad [Q_2, T^+] = 0 \quad , \quad [Q_1, T^-] = 0 \quad , \quad [Q_2, T^-] = -Q_1 \quad , \quad (15)$$

$$[\bar{Q}_1, T^+] = 0 \quad , \quad [\bar{Q}_2, T^+] = \bar{Q}_1 \quad , \quad [\bar{Q}_1, T^-] = \bar{Q}_2 \quad , \quad [\bar{Q}_2, T^-] = 0 \quad ,$$

$$[Q_{1,2}, T^0] = \pm \frac{1}{2} Q_{1,2} \quad , \quad [\bar{Q}_{1,2}, T^0] = \mp \frac{1}{2} \bar{Q}_{1,2} \quad ,$$

$$[Q_{1,2}, J] = -\frac{1}{2} Q_{1,2} \quad , \quad [\bar{Q}_{1,2}, J] = \frac{1}{2} \bar{Q}_{1,2} \quad .$$

A differential representation of the algebra $osp(2, 2)$ can be realized by the following 2×2 differential-matrix operators:

$$T_n^+ = \begin{pmatrix} x^2 d_x - nx & 0 \\ 0 & x^2 d_x - (n+1)x \end{pmatrix} \quad ,$$

$$T_n^0 = \begin{pmatrix} x d_x - \frac{n}{2} & 0 \\ 0 & x d_x - \frac{n+1}{2} \end{pmatrix} \quad ,$$

$$T_n^- = \begin{pmatrix} d_x & 0 \\ 0 & d_x \end{pmatrix} \quad , \quad J_n = \begin{pmatrix} -\frac{n+2}{2} & 0 \\ 0 & -\frac{n+1}{2} \end{pmatrix} \quad , \quad (16)$$

$$Q_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \quad Q_2 = \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} ,$$

$$\bar{Q}_1 = \begin{pmatrix} 0 & xd_x - (n+1) \\ 0 & 0 \end{pmatrix} , \quad \bar{Q}_2 = \begin{pmatrix} 0 & -d_x \\ 0 & 0 \end{pmatrix} .$$

Here x is a real variable, $d_x \equiv d/dx$, and n is a real number. It is easily checked that these matrices satisfy the $osp(2,2)$ (anti-) commutator relations Eq.(15). For non-negative integer n , there exists for the $osp(2,2)$ algebra a $(2n+3)$ -dimensional representation, with an invariant subspace P_{n+1}^n consisting of two-component functions of the form

$$\psi(x) = \begin{pmatrix} q_n(x) \\ p_{n+1}(x) \end{pmatrix} , \quad (17)$$

where $q_n(x)$ and $p_{n+1}(x)$ are polynomials of degree n and $n+1$, respectively.

4 Hidden Lie-Algebraic Structure of The Dirac Equation

We now show that the $osp(2,2)$ superalgebra is indeed the hidden algebraic structure underlying the quasi-exact solvability of Eq.(12) when ϵ is a non-negative integer $n \geq 0$. First we rewrite Eq.(12) as

$$\begin{aligned} T_Q(x) Q_n(x) &= 0 , \\ T_Q(x) &\equiv (x^2 + x_0 x) \frac{d^2}{dx^2} + (-x^3 - x_0 x^2 + 2\beta x + 2\beta x_0) \frac{d}{dx} \\ &\quad + nx^2 + (nx_0 + b - c)x + bx_0 . \end{aligned} \quad (18)$$

We have put $\epsilon = n$, and write $Q(x)$ as $Q_n(x)$ to indicate that it is a polynomial of degree n .

This equation may be cast into the matrix form

$$\mathbf{T}_Q \begin{pmatrix} Q_n(x) \\ 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 \\ T_Q(x) & 0 \end{pmatrix} \begin{pmatrix} Q_n(x) \\ 0 \end{pmatrix} = 0 . \quad (19)$$

The operator $\mathbf{T}_Q(x)$ is the 2×2 matrix operator in Eq.(19). Written in this form, the superalgebraic structure hidden in it can be clearly exhibited. The 2×2 matrix operator \mathbf{T}_Q turns out to be expressible as a linear combination of the $osp(2, 2)$ generators in Eq.(16). The two-component wave function in Eq.(19) is simply an element in the invariant subspace P_{n+1}^n of the algebra with the lower component $p_{n+1}(x) = 0$ (the form of $p_{n+1}(x)$ is immaterial here, as \mathbf{T}_Q annihilates the lower component of any element in the subspace P_{n+1}^n). We shall demonstrate this below.

First let us express the various terms in \mathbf{T}_Q in terms of the generators in Eq.(16). After some algebras, we obtain the following results:

$$2Q_2T_n^0T_n^- - Q_1T_n^+T_n^- = \begin{pmatrix} 0 & 0 \\ x^2d_x^2 & 0 \end{pmatrix} ,$$

$$Q_2T_n^-T_n^- = \begin{pmatrix} 0 & 0 \\ xd_x^2 & 0 \end{pmatrix} , \quad Q_2T_n^+ = \begin{pmatrix} 0 & 0 \\ x^3d_x - nx^2 & 0 \end{pmatrix} ,$$

$$2Q_2T_n^0 - Q_1T_n^+ = \begin{pmatrix} 0 & 0 \\ x^2d_x & 0 \end{pmatrix} ,$$

$$2(Q_2T_n^0 - Q_1T_n^+) = \begin{pmatrix} 0 & 0 \\ nx & 0 \end{pmatrix} ,$$

$$Q_2T_n^- = \begin{pmatrix} 0 & 0 \\ xd_x & 0 \end{pmatrix} , \quad Q_1T_n^- = \begin{pmatrix} 0 & 0 \\ d_x & 0 \end{pmatrix} .$$

With these expressions, the operator \mathbf{T}_Q can then be written as

$$\begin{aligned} \mathbf{T}_Q = & (2Q_2T_n^0T_n^- - Q_1T_n^+T_n^-) + x_0Q_2T_n^-T_n^- - Q_2T_n^+ \\ & - x_0(2Q_2T_n^0 - Q_1T_n^+) + 2\beta(Q_2T_n^- + x_0Q_1T_n^-) \\ & + 2x_0(Q_2T_n^0 - Q_1T_n^+) + (b - c)Q_2 + bx_0Q_1 . \end{aligned} \quad (20)$$

We have therefore succeeded in expressing \mathbf{T}_Q as a linear combination of the generators of $osp(2, 2)$ with finite dimensional subspace. The underlying algebraic structure responsible for the quasi-exact solvability of Eq.(12) is thus demonstrated.

The underlying algebraic structure responsible for the quasi-exact solvability of Eq.(13) can be obtained in the same way. In fact, as mentioned in Sect. 2, Eq.(13) is in the same form as Eq.(12), only with the parameters x , b , c and n being replaced by x' , b' , c' and $n+1$, respectively. Hence, we immediately see that the corresponding operator \mathbf{T}_P will be of exactly the same form as \mathbf{T}_Q with the same replacements of the corresponding parameters.

5 Summary

In this paper we have unveiled the underlying symmetry responsible for the quasi-exact solvability of planar Dirac electron in a Coulomb and a magnetic field. The relevant second order differential operator acting on any one component of the two-component wave function was recast in a 2×2 matrix form. This 2×2 differential-matrix operator was then shown to be expressible as a linear combination of the generators of the superalgebra $osp(2, 2)$, thus exhibiting the algebraic structure of the QES Dirac system. With this result, all the algebraic structures making the systems of planar charged particles in Coulomb and magnetic fields QES have been identified.

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